TRANSFORMATIONS OF COMPACT LOCALLY CONFORMALLY KÄHLER MANIFOLDS

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Abstract. We characterize compact locally conformally Kähler (l.c.K.) manifolds by means of the existence of a purely conformal, holomorphic circle action. As an application, we determine the structure of the compact locally conformally Kähler manifolds with parallel Lee form. We introduce the Lee-Cauchy-Riemann (LCR) transformations as a class of diffeomorphisms preserving the specific G-structure of l.c.K. manifolds. Then we characterize the Hopf manifolds, up to holomorphic isometry, as compact l.c.K. manifolds admitting a certain closed LCR action of $\mathbb{C}^*$.

1. Introduction

Let (M, g, J) be a connected, complex Hermitian manifold of complex dimension $n \geq 2$. We denote its fundamental two-form ω defined by $\omega(X, Y) = g(X, JY)$.

Definition 1.1. If ω satisfies the integrability condition

$$d\omega = \theta \wedge \omega \quad \text{with } d\theta = 0$$

the manifold is called *locally conformally Kähler* (l.c.K.).

The closed one-form θ is called the Lee form and it encodes the geometric properties of such a manifold (see [3] and the bibliography therein). Note that in complex dimension at least 3, if $d\omega = \theta \wedge \omega$, then θ is automatically closed. The second condition in the definition is necessary only on complex surfaces. Let ∇^g be the Levi-Civita connection of the l.c.K. metric g. The Weyl connection

$$D = \nabla^g - \frac{1}{2}\{\theta \otimes Id + Id \otimes \theta - g \otimes \theta^\sharp\}$$

is almost complex ($DJ = 0$) and preserves the conformal class of g ($Dg = \theta \otimes g$). Hence, a locally conformally Kähler manifold is a

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Hermitian-Weyl manifold. The converse is also true in complex dimension at least 6 (see [9]).

A locally conformally Kähler manifold (M, g, J) whose Lee form is ∇^g-parallel is called a *Vaisman manifold*. This subclass was introduced by I. Vaisman under the name of generalized Hopf manifolds; but not all of the Hopf surfaces have parallel Lee form, cf. [1], [4]. The structure of compact Vaisman manifolds is better understood. Here we recall some facts. At the topological level, it is known that b_1 is odd ([13]). Moreover, the Lee field $\theta^\sharp$ is real analytic ($\mathcal{L}_{\theta^\sharp} J = 0$) and g-Killing. The leaves of the foliation Null θ have an induced Sasakian structure. We note that $\theta^\sharp$ and $J\theta^\sharp$ generate a 2-dimensional, holomorphic, integrable distribution whose orthogonal distribution is preserved by J. When $\theta^\sharp$ generates an S^1 action by g-isometries, its action is quasi-regular (i.e., without fixed points), and so the quotient space is a Sasakian orbifold whose characteristic field is the projection of $J\theta^\sharp$. The Riemannian universal covering space $\tilde{M}$ of M then splits as $\mathbb{R} \times \tilde{N}$ where $\tilde{N}$ is a Sasakian manifold (as concerns Sasakian manifolds, see [2]). An immediate example (see [12]) is the Hopf manifold $\mathbb{C}^n - \{0\}/\Gamma$, where Γ is the cyclic group generated by $(z^i) \mapsto (2z^i)$, endowed with the projection of the metric $(\sum |z^i|^2)^{-1} \sum dz^i \otimes d\overline{z}^i$, globally conformal with the standard flat metric of $\mathbb{C}^n$. As the Hopf manifold is diffeomorphic with $S^1 \times S^{2n-1}$ it cannot bear any Kähler metric. Moreover, the Lee form of this metric is easily seen to be parallel with respect to the Levi-Civita connection.

In this paper we try to understand the influence of the structure of some transformations groups on the geometry of a compact l.c.K. manifold.

We first consider $Aut_{l.c.K.}(M)$, the (compact) group of all conformal, holomorphic diffeomorphisms. Some of its elements may have the stronger property to not restrict to a local isometry on any chart of the l.c.K. structure; we call them *purely conformal*. If a circle in $Aut_{l.c.K.}(M)$ is purely conformal, its lift to the universal cover is an $\mathbb{R}$ action (cf. Lemma 2.1). As a consequence, a compact, semi-simple subgroup of $Aut_{l.c.K.}(M)$ cannot contain purely conformal transformations (Corollary (2.2)). Our first main result (Theorem A) states that the existence of a purely conformal circle in $Aut_{l.c.K.}(M)$ assures the existence of a metric with parallel Lee form in the given conformal class. Note that, till now, l.c.K. manifolds with parallel Lee form has been characterized in the l.c.K. class by means of second order curvature conditions (involving the Ricci curvature) or, when the Lee field is regular, as circle bundles over Sasakian manifolds (as described above). From this result we derive the structure theorem for Vaisman manifolds

(without any assumption on the regularity of its natural foliations). Roughly speaking, a Vaisman manifold is isometric with $S^1 \underset{Q}{\times} W$ where $S^1 \to M \xrightarrow{\pi} W/Q$ is a Seifert fiber space over a Sasakian orbifold W/Q (see Corollary B).

In the second half of the paper we introduce a larger group of diffeomorphisms, containing the l.c.K.-ones, the *Lee-Cauchy-Riemann* (LCR) transformations. These are characterized as preserving the specific G-structure of a l.c.K. manifold. Precisely, with respect to an orthonormal coframe $\{\theta, \theta\circ J, \theta^\alpha, \overline{\theta}^\alpha\}_{\alpha=1,\cdots,n-1}$ adapted to a l.c.K. manifold (M, g, J), a LCR transformation f acts as:

$$f^*\theta = \theta, \quad f^*(\theta\circ J) = \lambda\cdot(\theta\circ J),$$
$$f^*\theta^\alpha = \sqrt{\lambda}\cdot\theta^\beta U^\alpha_\beta + (\theta\circ J)\cdot v^\alpha,$$
$$f^*\bar{\theta}^\alpha = \sqrt{\lambda}\cdot\bar{\theta}^\beta \overline{U}^\alpha_\beta + (\theta\circ J)\cdot \overline{v}^\alpha,$$

λ being a positive, smooth function and U^α_β a matrix in $\mathrm{U}(n-1)$. The main results of this part (Theorem C and Corollary D), essentially say that a compact l.c.K. manifold with the Lee field generated by a circle action and which admits a closed, non-compact flow of LCR-transformations is bi-holomorphically isometric with a finite quotient of a Hopf manifold. The interest (and difficulty) of this statement lies in the consideration of a non-compact subgroup of the group of LCR transformations. The proof relies heavily on techniques specific to CR geometry. It is to be noted that, to the authors' knowledge, the only previous attempts to characterize the Hopf manifolds among the Vaisman manifolds used curvature (precisely: conformally flatness) or spectral properties (see [3]).

2. Locally conformally Kähler transformations

Let (M, J) be a complex manifold. A l.c.K. structure on (M, J) is a local family $\{U_\alpha, g_\alpha\}_{\alpha\in\Lambda}$ where $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover and g_α is a Kähler metric on each U_α. On non-empty intersections $U_\alpha\cap U_\beta$, there exist positive constants $\lambda_{\beta\alpha}$ such that $g_\beta = \lambda_{\beta\alpha} g_\alpha$. It turns out that $\{\lambda_{\beta\alpha}\}$ is a 1-cocycle on M. Viewed $H^1(M;\mathbb{R}^+)$ as the sheaf cohomology of locally defined smooth positive functions, there exists a local family $\{f_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ consisting of smooth positive functions defined on each neighborhood such that $\delta^0 f(\alpha,\beta) = \dfrac{f_\alpha}{f_\beta} = \lambda_{\beta\alpha}$ on $U_\alpha\cap U_\beta \neq \emptyset$. Setting $g|U_\alpha = f_\alpha\cdot g_\alpha$, there exists a Hermitian metric g on M which is locally conformal to Kähler metrics.

Definition 2.1. Two l.c.K. structures $\{U_\alpha, g_\alpha\}_{\alpha\in\Lambda}$, $\{U_\alpha, {g'}_\alpha\}_{\alpha\in\Lambda}$ are equivalent if there exists a local family $\{c_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ (more precisely, a refinement) where each c_α is a constant number such that ${g'}_\alpha = c_\alpha \cdot g_\alpha$.

Given a l.c.K. manifold (M, g, J), since the Lee form θ is exact locally, there exists an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and positive functions f_α defined on U_α such that $df_\alpha = \theta|U_\alpha$. It follows that the local metrics defined on U_α by $g_\alpha = e^{-f_\alpha}\cdot g|U_\alpha$ are Kähler and conformally related: $e^{f_\beta}\cdot g_\beta = e^{f_\alpha}\cdot g_\alpha$ on non-empty intersections $U_\alpha \cap U_\beta$. Consequently, $\lambda_{\beta\alpha} = e^{f_\alpha - f_\beta}$ are positive constants satisfying the cocycle condition. Hence there is a locally conformally Kähler structure $\{U_\alpha, g_\alpha\}_{\alpha\in\Lambda}$ adapted to each l.c.K. manifold (M, g, J).

Proposition 2.1. *The set of equivalence classes of l.c.K. structures on a complex manifold (M, J) is in one-to-one correspondence with the set of conformal classes of l.c.K. metrics on (M, J).*

Proof. Supose that a l.c.K. structure $\{U_\alpha, g_\alpha\}_{\alpha\in\Lambda}$ is equivalent to another $\{U_\alpha, g'_\alpha\}_{\alpha\in\Lambda}$. By the definition, $g'_\alpha = c_\alpha \cdot g_\alpha$. As $g'_\beta = {\lambda'}_{\beta\alpha} g'_\alpha$, $g_\beta = \lambda_{\beta\alpha} g_\alpha$, we have $\lambda'_{\beta\alpha} = c_\beta \cdot \lambda_{\beta\alpha} \cdot c_\alpha^{-1}$ on the intersection $U_\alpha \cap U_\beta$. Let $\delta^0 f(\alpha,\beta) = \dfrac{f_\alpha}{f_\beta} = \lambda_{\beta\alpha}$ on $U_\alpha \cap U_\beta \neq \emptyset$. Similarly for $\lambda'_{\beta\alpha} = \dfrac{f'_\alpha}{f'_\beta}$ as above. Thus there is a global function τ on M such that $\tau|U_\alpha = c_\alpha f'_\alpha f_\alpha^{-1}$. By the definition, the l.c.K. metrics g, g' satisy $g|U_\alpha = f_\alpha \cdot g_\alpha$, $g'|U_\alpha = f'_\alpha \cdot g'_\alpha$. Hence $\tau \cdot g|U_\alpha = g'|U_\alpha$ for each α. Thus the equivalence class of $\{U_\alpha, g_\alpha\}_{\alpha\in\Lambda}$ determines the conformal class $[g]$.

Conversely, if $g' = \lambda \cdot g$ for a l.c.K. metric g where λ is some positive function, note that the fundamental two-form ω' satisfies $d\omega' = \theta' \wedge \omega'$ where $\theta' = \theta + d\log\lambda$. As $d\omega' = 0$, (M, g', J) is also a l.c.K. manifold. Let $df_\alpha = \theta|U_\alpha$ and $df'_\alpha = \theta'|U_\alpha$ as above. Since $df'_\alpha = \theta'|U_\alpha = \theta|U_\alpha + d\log\lambda$, there is some constant c_α such that $\log c_\alpha + f'_\alpha = f_\alpha + \log\lambda$ on U_α. In particular, $e^{f'_\alpha} e^{-f_\alpha} = \lambda c_\alpha^{-1}$. By the defintion, $g'_\alpha = e^{-f'_\alpha}\cdot g' = e^{-f'_\alpha}\cdot c_\alpha \cdot g = c_\alpha \cdot g_\alpha$. So the induced l.c.K. structures $\{U_\alpha, g_\alpha\}_{\alpha\in\Lambda}$, $\{U_\alpha, {g'}_\alpha\}_{\alpha\in\Lambda}$ are equivalent. □

Let us denote $Aut_{l.c.K.}(M)$ the group of l.c.K. transformations: it consists of all diffeomorphisms of M preserving the l.c.K. structure adapted to g. Explicitly, $\varphi \in Aut_{l.c.K.}(M)$ satisfies the conditions:

$$\varphi_* J = J\varphi_*$$
$$\varphi^* g_\beta = \mu_{\beta\alpha} g_\alpha, \quad \mu_{\beta\alpha} = ct. > 0 \quad \text{whenever } \varphi(U_\alpha) \subseteq U_\beta$$

As regards $Aut_{l.c.K.}(M)$, the following result was proved by the first named author in [6].

Proposition 2.2. *Let (M, g, J) be a locally conformally Kähler manifold of complex dimension $n \geq 2$. Then $Aut_{l.c.K.}(M)$ is a closed subgroup of the group of all conformal diffeomorphisms of (M, g). Moreover, if M is compact, $Aut_{l.c.K.}(M)$ is a compact Lie group.*

By the definition, the first statement is clear. As for the second one, if $Aut_{l.c.K.}(M)$ is noncompact, then also the group of all conformal transformations of (M, g) is noncompact. Then, by the celebrated result of Obata and Lelong-Ferrand, (M, g) is conformally equivalent with the sphere S^{2n}, $n \geq 2$. Hence M is simply connected, thus Kählerian. It is well known that a conformally flat Kähler manifold of dimension $n \geq 3$ is flat, which is impossible, while S^4 has no complex structure.

Since a l.c.K. manifold (M, g, J) is compact in our case, $Aut_{l.c.K.}(M)$ is a compact Lie group. Averaging the metric g by the compact group, we obtain a $Aut_{l.c.K.}(M)$-invariant metric g' conformal to g.

Proposition 2.3. *Given a l.c.K. structure on a compact complex manifold (M, J), there exists a l.c.K. metric (M, g, J) for which the group $Aut_{l.c.K.}(M)$ acts as isometries. Especially, $Aut_{l.c.K.}(M)$ leaves invariant the Lee form θ and the anti-Lee form $\theta \circ J$.*

We are interested in studying the possible subgroups of $Aut_{l.c.K.}(M)$. It is necessary to distinguish a special class of conformal transformations:

Definition 2.2. An element $a \in Aut_{l.c.K.}(M)$ is called *purely conformal* if $a^* g_\alpha = c_{\alpha\beta} g_\beta$ with $c_{\alpha\beta} \neq 1$ for all α, β (*i.e.* a is not a local isometry with respect to any of the local Kähler metrics). A subgroup of $Aut_{l.c.K.}(M)$ is called *purely conformal* if it contains only purely conformal holomorphic transformations.

Remark 2.1. On any compact locally conformally Kähler manifold with parallel Lee form, if the Lee field generates a circle action, this is purely conformal. We shall see this phenomenon more rigorously in Corollary B. (See also [3], [13].) Such examples of l.c.K. manifolds with regular (respectively quasi-regular) parallel Lee field are principal circle bundles over compact Hodge manifolds (respectively Brieskorn manifolds of a certain type).

In order to see the action of the fundamental group, we are looking at the universal covering space $\tilde{M}$ rather than to M itself in this paper. Let $(\tilde{M}, \tilde{g})$ be the universal Riemannian covering space of (M, g) and denote equally with J the lifted complex structure. Then the lifted Lee form $\tilde{\theta}$ is exact (as $\tilde{M}$ is simply connected): $\tilde{\theta} = d\tau$. Thus $(\tilde{M}, \tilde{g}, J)$ is

globally conformally Kähler. Let h be the Kähler metric $h = e^{-\tau} \cdot \tilde{g}$. Locally, on the inverse image $\tilde{U}_\alpha$ of U_α, one has

$$h = e^{-\tau} \cdot \tilde{f}_\alpha \tilde{g}_\alpha \tag{2.1}$$

Note that, as h and $\tilde{g}_\alpha$ are Kähler metrics, the function $e^{-\tau} \cdot \tilde{f}_\alpha$ is a constant. Clearly $\pi_1(M)$ acts by holomorphic $\tilde{g}$-isometries, hence by holomorphic conformal transformations with respect to h. But it is well-known that a holomorphic conformal transformation of a Kähler manifold is homothetic. The converse costruction is also possible. We obtain:

Proposition 2.4. ([13]) *Let (M, J) be a complex manifold and $\tilde{M}$ its universal Riemannian cover. Then M admits a locally conformally Kähler structure if and only if $\tilde{M}$ admits a Kählerian structure with respect to which $\pi_1(M)$ acts by homothetic holomorphic transformations.*

The next lemma provides a characterization of purely conformal l.c.K. transformations:

Lemma 2.1. *A locally conformally Kähler circle action on M lifts to an $\mathbb{R}$-action by holomorphic, non-trivial h-homotheties on $\tilde{M}$ if and only if it is purely conformal.*

Proof. A l.c.K. S^1 action lifts to an S^1 or a $\mathbb{R}$-action by homotheties on $\tilde{M}$. Let us show that, owing to the purely conformal character of S^1 in $Aut_{l.c.K.}(M)$, it is impossible to lift to an S^1-action. Let $b \in S^1$ be such that, for a given α, $b(U_\alpha) \subseteq U_\alpha$ (such a b must be small enough, close to 1.) Then $b^* g_\alpha = c_\alpha \cdot g_\alpha$ with $c_\alpha = const. \neq 1$ on U_α. This b lifts to a $\tilde{b}$ acting on $\tilde{M}$ and we have $\tilde{b}^* \tilde{g}_\alpha = c_\alpha \cdot \tilde{g}_\alpha$ on $\tilde{U}_\alpha$. The Kähler metric h is related to the lifted globally conformally Kähler metric $\tilde{g}$ by $h = e^{-\tau} \tilde{g}$, hence (2.1) implies (recall that $e^{-\tau} \cdot \tilde{f}_\alpha$ is constant on U_α):

$$\tilde{b}^* h = (e^{-\tau} \cdot \tilde{f}_\alpha) \cdot \tilde{b}^* \tilde{g}_\alpha = (e^{-\tau} \cdot \tilde{f}_\alpha) \cdot c_\alpha \tilde{g}_\alpha.$$

Now, if $\tilde{b}$ is an isometry, then $c_\alpha = 1$, contradiction. Thus, if ρ is the homomorphism defined on the set of all h-homotheties of $\tilde{M}$ with values in $\mathbb{R}^+$ by the formula $t^* h = \rho(t) h$, we derive $\rho(\tilde{b}) \neq 1$. Hence, the lift of S^1 cannot be S^1, otherwise its image by ρ would be a compact *non-trivial* subgroup of $\mathbb{R}^+$, contradiction. □

If $g' = \lambda \cdot g$, then the Kähler metric is obtained as $h' = e^{\tau'} \cdot \tilde{g}'$ where $\tilde{\theta}' = d\tau'$. As h, h' are Kähler, we have $h' = c' \cdot h$ where $c' = e^{-\tau'} \tilde{\lambda} e^{\tau}$ is a constant. So the lifted group $\mathbb{R}$ consists of nontrivial h-homotheties if and only if it consists of nontrivial h'-homotheties.

Corollary 2.1. *A purely conformal circle action does not depend on the conformal class of l.c.K. metrics* $[g]$.

The following result imposes a restriction on the existence of purely conformal circle actions of a l.c.K. manifold:

Corollary 2.2. *A compact semisimple subgroup of* $Aut_{l.c.K.}(M)$ *cannot contain any purely conformal transformation.*

Proof. Suppose, *ad absurdum*, that M admits a purely conformal S^1 action. According to Lemma 2.1 (and to its proof), this action lifts to an $\mathbb{R}$-action by h-homotheties and ρ is injective. But ρ can be understood both as being defined on $\mathbb{R}$ and on $\pi_1(M)$, because $\mathbb{R}$ and $\pi_1(M)$ are both contained in the group of h-homotheties of $\tilde{M}$. Further, for a fixed $x \in M$, consider the evaluation map $ev : G \to M$, $ev(a) = ax$. We denote also by ev the restriction of this map to the considered S^1, so obtain the commutative diagram:

$$\begin{tikzcd} G \arrow[r, "ev"] & M \\ S^1 \arrow[u] \arrow[ur, "ev"'] & \end{tikzcd}$$

where the vertical arrow is injective. Taking into account the previous observation, at the homotopy level we then have the commutative diagram:

$$\begin{tikzcd} \pi_1(G) \arrow[r, "ev_\sharp"] & \pi_1(M) \arrow[r, "\rho"] & \mathbb{R}^+ \\ \mathbb{Z} \arrow[u] \arrow[ur, "ev_\sharp"'] \arrow[urr, "\rho"'] & & \end{tikzcd}$$

Hence, $\rho \circ ev_\sharp(\mathbb{Z})$ is infinite in $\mathbb{R}^+$. But, G being semi-simple, it has no torus as a direct summand and has finite $\pi_1(G)$. Thus, chasing on the upper side of the diagram, the image of $\mathbb{Z}$ in $\mathbb{R}^+$ through $\rho \circ ev_\sharp$ is finite. This contradiction completes the proof. $\square$

Our first main result shows that the existence of purely conformal circles in $Aut_{l.c.K.}(M)$ characterizes the existence of metrics with parallel Lee form in a given conformal class:

Theorem A. *Let* (M, g, J) *be a compact locally conformally Kähler, non-Kähler manifold. If* $Aut_{l.c.K.}(M)$ *contains a purely conformal subgroup* S^1*, then there exists a metric with parallel Lee form in the conformal class of* g.

Proof of Theorem A.

Let us call ξ the generating vector field of the $\mathbb{R}$-action on $\tilde{M}$ and let $< \varphi_t >$ be the one-parameter subgroup generated by ξ. Using the fact that φ_t are holomorphic maps we derive

$$(2.2)\qquad [\xi, J\xi] = \lim_{t\to 0} \frac{\varphi_{t*}(J\xi) - J\xi}{t} = \lim_{t\to 0} \frac{J\varphi_{t*}(\xi) - J\xi}{t} = 0,$$

thus the one-parameter subgroups generated by ξ and $J\xi$ commute.

Lemma 2.2. *The one-parameter subgroup $< \psi_t >$ generated by $J\xi$ is global.*

Proof. As the deck group of the covering π preserves ξ and commutes with J, $J\xi$ projects to a one-parameter subgroup on M. This one is global, by compactness of M and lifts to a global one-parameter subgroup on $\tilde{M}$ which clearly coincides with ψ_t. □

Remark 2.2. As ξ is a real analytic vector field, so is $J\xi$ (cf. [7], p. 76).

Let Ω be the Kähler form of the Kähler metric h on $\tilde{M}$. (Note that $\Omega = e^{-\tau} \cdot \tilde{\omega}$.) Because $\mathbb{R}$ acts by h-homotheties, we can write:

$$(2.3)\qquad \varphi_t^*\Omega = \rho(t)\cdot\Omega, \quad t \in \mathbb{R},\ \rho(t) \in \mathbb{R}^+.$$

As ρ is a nontrivial, continuous homomorphism, $\rho(t) = e^{at}$ for some constant $a \neq 1$. We may normalize $a = 1$ so that (2.3) is described as

$$(2.4)\qquad \varphi_t^*\Omega = e^t\cdot\Omega, \quad t \in \mathbb{R}.$$

Lemma 2.3. *Let $s : \tilde{M} \to \mathbb{R}$ be the smooth map defined as $s(x) = \Omega(J\xi_x, \xi_x)$. Then 1 is a regular value of s, hence $s^{-1}(1)$ is a codimension 1, smooth submanifold of $\tilde{M}$.*

Proof. Note that

$$\begin{aligned} s(\varphi_t x) &= \Omega(J\xi_{\varphi_t x}, \xi_{\varphi_t x}) = \Omega(\varphi_{t*}J\xi_x, \varphi_{t*}\xi_x) \text{ by (2.2)} \\ &= e^t\Omega(J\xi_x, \xi_x) = e^t s(x) \text{ by (2.4)} \end{aligned}$$

As $s(x) \neq 0$, $s^{-1}(1) \neq \emptyset$. On the other hand, from

$$\mathcal{L}_\xi\Omega = \lim_{t\to 0} \frac{\varphi_t^*\Omega - \Omega}{t} = \Omega,$$

we obtain for $x \in s^{-1}(1)$:

$$ds(\xi_x) = \xi s(x) = (\mathcal{L}_\xi s)(x) = (\mathcal{L}_\xi \Omega(J\xi, \xi))(x) = \Omega(J\xi_x, \xi_x) = s(x) = 1.$$

This proves that $ds : T_x\tilde{M} \to \mathbb{R}$ is onto and $s^{-1}(1)$ is a codimension-1, smooth submanifold of $\tilde{M}$.

□

Let now $W = s^{-1}(1)$. We can prove:

Lemma 2.4. *W is connected. The evaluation map $H : \mathbb{R} \times W \to \tilde{M}$, defined by $H(t, w) = \varphi_t w$ is an equivariant diffeomorphism.*

Proof. Let W_0 be a component of $s^{-1}(1)$ and let $\mathbb{R} \cdot W_0$ be the set $\{\varphi_t w \; ; \; w \in W_0, t \in \mathbb{R}\}$. As $\mathbb{R}$ acts freely and $s(\varphi_t x) = e^t s(x)$, we have $\varphi_t W_0 \cap W_0 = \emptyset$ for $t \neq 0$. Thus $\mathbb{R} \cdot W_0$ is an open subset of $\tilde{M}$. We now prove that it is also closed. Let $\overline{\mathbb{R} \cdot W_0}$ be the closure of $\mathbb{R} \cdot W_0$ in $\tilde{M}$. We choose a limit point $p = \lim \varphi_{t_i} w_i \in \overline{\mathbb{R} \cdot W_0}$. Then $s(p) = \lim s(\varphi_{t_i} w_i) = \lim e^{t_i} s(w_i) = \lim e^{t_i}$. We denote $t = \log s(p)$, $t = \lim t_i$, so $\varphi_t^{-1}(p) = \lim \varphi_{t_i}^{-1}(\lim \varphi_{t_i} w_i) = \lim w_i$. Since $s^{-1}(1)$ is regular (*i.e.* closed with respect to the relative topology induced from $\tilde{M}$), so is its component W_0. Hence $\varphi_t^{-1} p \in W_0$. Therefore $p = \varphi_t(\varphi_t^{-1} p) \in \mathbb{R} \cdot W_0$, proving that $\mathbb{R} \cdot W_0$ is closed in $\tilde{M}$. In conclusion, $\mathbb{R} \cdot W_0 = \tilde{M}$. Now, if W_1 is another component of $s^{-1}(1)$, the same argument shows $\mathbb{R} \cdot W_1 = \tilde{M}$. Thus, $\mathbb{R} \cdot W_0 = \mathbb{R} \cdot W_1$. This implies $W_0 = W_1$, in other words W is connected. □

Lemma 2.5. *W has a contact form η for which $\pi_* J\xi$ is the Reeb (characteristic) field.*

Proof. Let $i : W \to \tilde{M}$ be the inclusion and $\pi : \tilde{M} \to W$ be the canonical projection. For $X_w \in T_w W$ we put

$$\eta_w(X_w) = e^{-t} \Omega(\tilde{X}_{\varphi_t w}, \xi_{\varphi_t w}) \tag{2.5}$$

where $\tilde{X}_{\varphi_t w} \in T_{\varphi_t w} \tilde{M}$ such that $\pi_* \tilde{X} = X$. As $\mathbb{R} \to \tilde{M} \xrightarrow{\pi} W$ is a fiber bundle with $T\mathbb{R} =< \xi >$, η is well-defined. To prove that η is a contact form, we first note that by the definition

$$\pi^* \eta = -e^{-t} \cdot \iota_\xi \Omega \text{ on each } T_{\varphi_t w} \tilde{M}. \tag{2.6}$$

Then

$$i^* \pi^* \eta = (\pi i)^* \eta = \eta = i^*(-e^{-t} \cdot \iota_\xi \Omega) = -i^* \iota_\xi \Omega \text{ on } W.$$

Now recall that $\Omega = \mathcal{L}_\xi \Omega = d\iota_\xi \Omega + \iota_\xi d\Omega = d\iota_\xi \Omega$, thus

$$d\eta = -di^* \iota_\xi \Omega = -i^* d\iota_\xi \Omega = -i^* \Omega$$

and, moreover,

$$\eta(\pi_*(J\xi_w)) = \Omega(J\xi_w, \xi_w) = s(w) = 1 \text{ by the definition.}$$

Hence, $\eta \wedge d\eta^{n-1} \neq 0$ on W showing that η is a contact form.

Let us now show that $\pi_* J\xi$ is the characteristic field of η. For any distribution D on $\tilde{M}$, denote $D^\perp$ the orthogonal distribution with respect to the metric h. Then $\pi_* : \xi^\perp \to TW$ is an isomorphism and induces

an isomorphism $\pi_* : \{\xi, J\xi\}^\perp \to \mathrm{Null}\,\eta$. So, $\Omega(\tilde{X}, \xi) = h(\tilde{X}, J\xi) = 0$, $\Omega(\tilde{X}, J\xi) = h(\tilde{X}, -\xi) = 0$ for $\tilde{X} \in \{\xi, J\xi\}^\perp$. We now show that $d\eta(\pi_* J\xi, X) = 0$ for any $X \in \mathrm{Null}\,\eta$. We have $2d\eta(\pi_* J\xi, X) = -\eta([\pi_* J\xi, X])$. Let $\pi_* \tilde{X} = X$ for some $\tilde{X} \in \{\xi, J\xi\}^\perp$. Then, using (2.6) on W (*i.e.* for $t = 0$):

$$2d\eta(\pi_* J\xi, X) = -\pi^*\eta([J\xi, \tilde{X}]) = \iota_\xi \Omega([J\xi, \tilde{X}]) = \Omega(\xi, [J\xi, \tilde{X}]).$$

But $[\tilde{X}, \xi] = \lim_{t \to 0} \dfrac{1}{t}(\tilde{X} - \varphi_{-t*}\tilde{X})$ and

$$\Omega(\varphi_{-t*}\tilde{X}, J\xi) = h(\varphi_{-t*}\tilde{X}, \xi) = e^{-t} h(\tilde{X}, \varphi_{t*}\xi) = e^{-t} h(\tilde{X}, \xi) = 0$$

because $\tilde{X} \in \{\xi, J\xi\}^\perp$. So:

$$\Omega([\tilde{X}, \xi], J\xi) = \lim_{t \to 0} \frac{\Omega(\tilde{X}, J\xi) - \Omega(\varphi_{-t*}\tilde{X}, J\xi)}{t} = 0.$$

Since

$$\begin{aligned} 3d\Omega(\tilde{X}, \xi, J\xi) &= \tilde{X}\Omega(\xi, J\xi) - \xi\Omega(\tilde{X}, J\xi) + (J\xi)\Omega(\tilde{X}, \xi) \\ &\quad - \Omega([\tilde{X}, \xi], J\xi) - \Omega([\xi, J\xi], \tilde{X}) - \Omega([J\xi, \tilde{X}], \xi) = 0, \end{aligned}$$

we have

$$\tilde{X}\Omega(\xi, J\xi) - \Omega([J\xi, \tilde{X}], \xi) = 0.$$

If $\exp : T_w\tilde{M} \to \tilde{M}$ is the exponential map with respect to h, then the subset $\{t,\ \exp(\xi^\perp)\}$ constitutes a coordinate neighborhood of $\tilde{M}$ at w. In particular, the above $\tilde{X}$ is a linear combination of coordinate vector fields around w without containing $\frac{\partial}{\partial t}$. On the other hand, noting $\Omega(J\xi_{\varphi_t w}, \xi_{\varphi_t w}) = s(\varphi_t w) = e^t$ on $\tilde{M}$ from (2.3), it follows $\tilde{X}\Omega(\xi, J\xi) = -\tilde{X} \cdot e^t = 0$. We finally deduce $\Omega([J\xi, \tilde{X}], \xi) = 0$ on W and the proof of the lemma is complete.

□

It is now easy to verify the following:

Corollary 2.3. *$(W, i^*h, \pi_* J\xi)$ is a Sasakian manifold. In particular, if we denote by ϕ the tangent part of J to W (hence $\phi = \nabla^{h'}\pi_* J\xi$), we can say that (η, ϕ) is a pseudo-Hermitian structure on W.*

We have derived from (2.6) that

$$d(e^t \pi^* \eta) = -\Omega \text{ on } \tilde{M}. \tag{2.7}$$

For the given l.c.K. metric g, the Kähler metric h is obtained as $h = e^{-\tau} \cdot \tilde{g}$ where $d\tau = \tilde{\theta}$. (See (2.1).) As ω is the fundamental two-form of g, note that $\Omega = e^{-\tau} \cdot \tilde{\omega}$.

Put

$$\bar{\Theta} = -e^{-t} \cdot d(e^t \pi^* \eta) \ (= +e^{-t} \cdot \Omega). \tag{2.8}$$

Then the Hermitian metric $\bar{g}(X,Y) = \bar{\Theta}(JX,Y)$ is a l.c.K. metric of $\mathbb{R} \times W$ on which $\mathbb{R}$ acts as isometries. It is obvious that $\bar{g}$ has the parallel Lee form $-dt$. We obtain that

$$\bar{\Theta} = \mu \cdot \tilde{\omega} \text{ (equivalently } \bar{g} = \mu \cdot \tilde{g}) \tag{2.9}$$

where $\mu = e^{-(t+\tau)} : \tilde{M} \to \mathbb{R}^+$ is a smooth map.

Lemma 2.6. *$\pi_1(M)$ acts by isometries of $\bar{g}$.*

Proof. We prove the following two facts:

1. $\gamma^* \pi^* \eta = \pi^* \eta$ for every $\gamma \in \pi_1(M)$.
2. $\gamma^* e^t = \rho(\gamma) \cdot e^t$ where $\rho : \pi_1(M) \to \mathbb{R}^+$ is a homomorphism similar to the case of $\mathbb{R} = \{\varphi_\theta\}$.

First note that as $\mathbb{R} = \{\varphi_\theta\}$ centralizes $\pi_1(M)$, $\gamma_* \xi = \xi$ for $\gamma \in \pi_1(M)$. Recall (cf. (2.5)) that $\pi_* : \xi^\perp \to TW$, $\pi_* : \{\xi, J\xi\}^\perp \to \text{Null } \eta$ are isomorphic. As $\pi_1(M)$ acts on $\tilde{M}$ as holomorphic homothetic transformations (cf. Proposition 2.4), $\pi_1(M)$ leaves $X \in \{\xi, J\xi\}^\perp$ invariant. If $X \in \{\xi, J\xi\}^\perp$, then $\gamma^* \pi^* \eta(X) = \eta(\pi_* \gamma_* X) = 0$. As $\pi_* J\xi$ is a characteristic vector field for η, $\gamma^* \pi^* \eta(J\xi) = \eta(\pi_* \gamma_* J\xi) = \eta(\pi_* J\xi) = 1$. This concludes that $\gamma^* \pi^* \eta = \pi^* \eta$. On the other hand, if we note $\gamma_* \xi = \xi$, then

$$\begin{aligned} \gamma^*(\iota_\xi \Omega)(X) &= \Omega(\xi, \gamma_* X) = \Omega(\gamma_* \xi, \gamma_* X) \\ &= \gamma^* \Omega(\xi, X) = \rho(\gamma) \cdot \Omega(\xi, X) \\ &= \rho(\gamma) \cdot \iota_\xi \Omega(X) \end{aligned}$$

where $\rho(\gamma)$ is a positive constant number. As $\gamma^* \pi^* \eta = \pi^* \eta$ from **1.** and $\pi^* \eta = -e^{-t} \cdot \iota_\xi \Omega$ from (2.6), we obtain that $\gamma^* e^{-t} \cdot \rho(\gamma) = e^{-t}$. Equivalently, $\gamma^* e^t = \rho(\gamma) \cdot e^t$. This shows **1** and **2**. Now, we prove (2.6).

From (2.8),

$$\begin{aligned} \gamma^* \bar{\Theta} &= \gamma^*(-e^{-t} \cdot d(e^t \pi^* \eta)) \\ &= -\rho(\gamma)^{-1} \cdot e^{-t} d(\rho(\gamma) \cdot e^t \gamma^* \pi^* \eta) \\ &= -e^{-t} \cdot d(e^t \pi^* \eta) = \bar{\Theta}. \end{aligned}$$

Since $\bar{g}(X,Y) = \bar{\Theta}(JX,Y)$ by the definition, $\pi_1(M)$ acts through holomorphic isometries of $\bar{g}$.

$\square$

From this lemma, the covering map $p : \tilde{M} \to M$ induces a l.c.K. metric $\hat{g}$ with parallel Lee form $\hat{\theta}$ on M such that $p^*\hat{g} = \bar{g}$ and $p^*\hat{\theta} = -dt$. Since $\tilde{g}$ is a lift of g to $\tilde{M}$, using the equation (2.9), we derive

$$\gamma^*\mu \cdot \gamma^*\tilde{g} = \gamma^*\mu \cdot \tilde{g} = \gamma^*\bar{g} = \bar{g} = \mu \cdot \tilde{g}$$

therefore $\gamma^*\mu = \mu$. Since μ factors through a map $\hat{\mu} : M \to \mathbb{R}^+$ so that $p^*\hat{g} = p^*(\hat{\mu} \cdot g)$, we have $\hat{\mu} \cdot g = \hat{g}$. The conformal class of g contains a l.c.K. metric with parallel Lee form.

This finishes the proof of Theorem A.

Using this theorem, as is noted in Remark 2.1, we can make clear the case when the parallel Lee field does not generate a (free) circle action. Equivalently, the parallel Lee field is not a quasi-regular field.

Corollary B. (The structure of Vaisman manifolds)**.** *Let (M, g, J) be a compact locally conformally Kähler, non-Kähler manifold with parallel Lee form θ.*

(I) *Suppose that the Lee field $\theta^{\#}$ does not generate a circle. Then there exists a l.c.K. metric $\hat{g}$ with paralell Lee form in the conformal class of g on (M, J) such that:*

1. *The paralell Lee field $\hat{\theta}^{\#}$ of $\hat{g}$ generates a circle action S^1.*
2. *$(M, \hat{g}, J)$ is isometric to the quasi-regular Vaisman manifold $S^1 \underset{Q}{\times} W$ where $S^1 \to M \xrightarrow{\hat{\pi}} W/Q$ is a Seifert fiber space over a Sasakian orbifold W/Q.*
3. *The Kähler form on the universal covering space $\tilde{M}$ is identified with $-d(t\pi^*\tilde{\eta})$ for which the contact form η is a Q-invariant pseudo-Hermitian structure on W. Here t is the coordinate for $\mathbb{R}^+$. The projection $\pi : \tilde{M} \to W$ is the lift of $\hat{\pi}$ to $\tilde{M} = \mathbb{R}^+ \times W$.*

(II) *Suppose that the Lee field $\theta^{\#}$ generates a circle S^1. Then (M, g, J) is itself of the quasi-regular form $S^1 \underset{Q}{\times} W$ for which the Kähler form on $\tilde{M}$ is $d(t\pi^*\tilde{\eta})$ and:*

1. *The paralell Lee form θ lifts to the form dt on $\tilde{M} = \mathbb{R}^+ \times W$ where t is the coordinate for $\mathbb{R}^+$.*
2. *The contact form $\tilde{\eta}$ is a Q-invariant Sasakian structure on W, i.e., $\pi^*\tilde{\eta}(X) = \tilde{g}(J\xi, X)$ for $X \in \xi^{\#}$.*

Proof. Let $\mathrm{ISO}(M, g)$ be the group of all holomorphic isometries of M. As the Lee field $\theta^{\#}$ is Killing, it generates a 1-parameter subgroup $\{\varphi_t\}_{t\in\mathbb{R}}$ of (holomorphic) isometries on M. Then $\{\varphi_t\}_{t\in\mathbb{R}} \subset \mathrm{ISO}(M, g)$. Since $\mathrm{ISO}(M, g)$ is compact, the closure $\mathcal{T}$ of $\{\varphi_t\}_{t\in\mathbb{R}}$ in $\mathrm{ISO}(M, g)$ is a k-torus ($k \geq 1$).

I. When the 1-parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ is not a circle, we can find a sequence of circles $\{S^1_i\}_{i\in\mathbb{N}}$ which approaches to $\{\varphi_t\}_{t\in\mathbb{R}}$. Denote by $\{\xi^i\}$ the vector field induced by S^1_i for each i. Then there exists ℓ with

$$\theta(\xi_\ell) \neq 0 \ \text{ everywhere in } M. \tag{2.10}$$

To prove this, suppose that for all i there exists a sequence of points $\{x_i\} \subset M$ such that $\theta_{x_i}(\xi_{x_i}) = 0$. As M is compact, there exists a point $x \in W$ such that $x = \lim_{i\to\infty} x_i$. If we note that ξ_{x_i} converges to $(\theta^{\#})_x$, then $0 = \lim_{i\to\infty} \theta(\xi_{x_i}) = \theta((\theta^{\#})_x) = g_x(\theta^{\#}, \theta^{\#})$, which is impossible.

Now, let $\{\phi_t\}_{t\in\mathbb{R}}$ (respectively $\tilde{\xi}_\ell$) be the lift of the circle S^1_ℓ (respectively ξ_ℓ) to $\tilde{M}$. Recall the Kähler form $\Omega = e^{-\tau}\cdot\tilde{\omega}$ on $\tilde{M}$ where $d\tau = \tilde{\theta}$ (cf. (2.1)). Note that each ϕ_t leaves invariant $\tilde{\omega}$ and so it satisfies:

$$\phi_t^*\Omega = e^{-(\phi_t^*\tau-\tau)}\cdot\Omega.$$

As Ω is Kähler, $e^{-(\phi_t^*\tau-\tau)}$ is constant on $\tilde{M}$. Suppose that every ϕ_t is an isometry, *i.e.* $\phi_t^*\tau - \tau = 0$ for all t. Then,

$$0 = \mathcal{L}_{\tilde{\xi}_\ell}(\tau) = d\tau(\tilde{\xi}_\ell) = \tilde{\theta}(\tilde{\xi}_\ell) = \theta(\xi_\ell) \neq 0 \text{ by (2.10)},$$

being a contradiction, and so some ϕ_t is a nontrivial homothety; $\phi_t^*\Omega = e^{ct}\cdot\Omega$ for a nonzeo constant c. Hence every ϕ_t is a nontrivial homothetic transformation. By Lemma 2.1, the circle S^1_ℓ is purely conformal. Then the results **1,2,3** of **I** follow from Theorem A.

We prove **II**. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a lift of S^1 to $\tilde{M}$ which induces a vector field ξ. Noting that φ_t is an isometry of $\tilde{g}$ because θ is parallel, we can write $\varphi_t^*\Omega = e^{-(\varphi_t^*\tau-\tau)}\cdot\Omega$ where $\Omega = e^{-tau}\cdot\tilde{\omega}$. Since $e^{-(\varphi_t^*\tau-\tau)}$ is constant on $\tilde{M}$ for each t, we may put $\tau\circ\varphi_t - \tau = c\cdot t$ for some constant c. By the definition $\theta(X) = g(\theta^{\#}, X)$, so is its lift $\tilde{\theta}(\tilde{X}) = \tilde{g}(\xi, \tilde{X})$ on $\tilde{M}$. As $\tilde{\theta} = d\tau$,

$$0 < \tilde{g}(\xi,\xi) = \tilde{\theta}(\xi) = \mathcal{L}_\xi\tau = \lim_{t\to 0}\frac{\tau\circ\varphi_t - \tau}{t} = c. \tag{2.11}$$

We may normalize $c = 1$ so that (cf. (2.3)):

$$\tau\circ\varphi_t(w) - \tau(w) = t \text{ for } \ w \in W, \tag{2.12}$$

$$\varphi_t^*\Omega = e^{-t}\cdot\Omega \text{ on}\tilde{M}. \tag{2.13}$$

In particular, the group $\{\varphi_t\}_{t\in\mathbb{R}}$ is isomorphic to $\mathbb{R}$. As in the proof of Theorem A, we have a contact form η on W. However, comparing (2.13) with (2.3), this will be defined as

$$\eta_w(X_w) = e^t\cdot\Omega(\tilde{X}_{\varphi_t w}, \xi_{\varphi_t w}) \text{ i.e., } \pi^*\eta = -e^t\cdot\iota_\xi\Omega, \tag{2.14}$$

where $\tilde{X}_{\varphi_t w} \in T_{\varphi_t w}\tilde{M}$ such that $\pi_*\tilde{X} = X$. (Compare (2.5).) Noting also that $\mathcal{L}_\xi\Omega = -\Omega$ in this case, we have that (cf. (2.7))

$$d(e^{-t} \cdot \pi^*\eta) = \Omega \text{ on } \tilde{M}. \tag{2.15}$$

On the other hand, we can define a 1-form $\tilde{\eta}$ on W (*that is, a Sasakian contact structure*, see [3]) by

$$\tilde{\eta}(X) = \tilde{g}(J\xi, \tilde{X})$$

for $\tilde{X} \in \xi^\perp$ with respect to $\tilde{g}$ such that $\pi_*\tilde{X} = X$. If we note $h = e^{-\tau} \cdot \tilde{g}$, then the distribution $\xi^\perp$ is the same as that of h. As $\pi_* : \xi^\perp \to TW$ is isomorphic, $\tilde{\eta}$ is well defined.

We compare these contact forms first. By $\Omega = e^{-t} \cdot \tilde{\omega}$ and (2.14), we compute

$$\pi^*\tilde{\eta}(\tilde{X}) = -\tilde{g}(\xi, \tilde{X}) = -\tilde{\omega}(\xi, \tilde{X}) = -e^\tau \cdot \Omega(\xi, \tilde{X}) = e^\tau \cdot (e^{-t} \cdot \pi^*\eta)(\tilde{X})$$

so that

$$\pi^*\tilde{\eta} = e^{\tau - t} \cdot \pi^*\eta \text{ on } \tilde{M}. \tag{2.16}$$

Moreover, $\varphi_s^*(\tau - t) = \tau \circ \varphi_s - (t + s) = \tau - t$ on W from (2.12). Thus there is a map $\nu : W \to \mathbb{R}$ with $\pi^* \circ \nu = \tau - t$. Hence, (2.16) implies:

$$\tilde{\eta} = e^\nu \cdot \eta \text{ on } W. \tag{2.17}$$

Finally, recall from Lemma 2.3 that $1 = s(w) = \Omega(J\xi_w, \xi_w) = e^{-\tau(w)} \cdot \tilde{\omega}(J\xi_w, \xi_w) = e^{-\tau(w)} \cdot \tilde{g}(\xi_w, \xi_w) = e^{-\tau(w)}$ because $c = 1$ in (2.11). Thus, from equation (2.12) we derive:

$$e^{\tau \circ \varphi_t(w)} = e^t \text{ for all } \varphi_t(w) \in \tilde{M}. \tag{2.18}$$

Using (2.18) and (2.15), (2.16)

$$\begin{aligned} \tilde{\omega} &= e^\tau \cdot \Omega = e^\tau \cdot d(e^{-t}\pi^*\eta) \\ &= e^\tau \cdot d(e^{-t}e^{t-\tau} \cdot \pi^*\tilde{\eta}) = e^\tau \cdot d(e^{-\tau} \cdot \pi^*\tilde{\eta}) \\ &= e^t \cdot d(e^{-t} \cdot \pi^*\tilde{\eta}) \end{aligned}$$

on $\tilde{M} = \mathbb{R}^+ \times W$. Therefore $\tilde{g}$ has the parallel Lee form dt such that $p^*\theta = dt$ for θ on M. This proves **II.**

□

Remark 2.3. **(1)** When a compact l.c.K. manifold (M, g, J) has the quasi-regular form $S^1 \times W/Q$ in the conformal class of g, it is still vague whether or not g itself has parallel Lee form.
(2) The Inoue surface S_N^- has a l.c.K. metric without parallel Lee form (see [11]). The associated Lee field $\theta^\#$ is Killing (without constant

norm) and generates an S^1-action by holomorphic isometries. It acts freely so that the orbit space is the 3-dimensional solvmanifold. On the other hand, a compact Sasakian manifold admits a nontrivial T^k-action ($k \geq 1$) generated by the Reeb field. If the Sasakian manifold is a closed aspherical manifold, then the fundamental group has a nontrivial center (at least, containing a free abelian group of rank k.) As the 3-dimensional solvmanifold has no center, it admits no Sasakian structure. Hence, this S^1 is not purely conformal. Note that the one-parameter subgroup of holomorphic transformations generated by $J\theta^{\#}$ is not a circle and it does not leave invariant the fundamental two-form.

3. Lee-Cauchy-Riemann transformations

In this section, we consider another kind of transformations of l.c.K. manifolds. As above, there exists an orthonormal, local coframe $\{\theta, \theta\circ J, \theta^\alpha, \overline{\theta}^\alpha\}_{\alpha=1,\cdots,n-1}$ adapted to a l.c.K. manifold (M, g, J). Here we have $=\omega|\{\theta^{\#}, J\theta^{\#}\}^\perp = \sum_1^{n-1}\delta_{\alpha\beta}\theta^\alpha\wedge\overline{\theta}^\beta$. Consider the diffeomorphism f which transforms:

$$\begin{aligned} &f^*\theta = \theta, \quad f^*(\theta\circ J) = \lambda\cdot(\theta\circ J), \\ &f^*\theta^\alpha = \sqrt{\lambda}\cdot\theta^\beta U^\alpha_\beta + (\theta\circ J)\cdot v^\alpha, \\ &f^*\bar{\theta}^\alpha = \sqrt{\lambda}\cdot\bar{\theta}^\beta\overline{U}^\alpha_\beta + (\theta\circ J)\cdot\overline{v}^\alpha \end{aligned} \tag{3.1}$$

for some positive, smooth function λ, and a matrix $U^\alpha_\beta \in \mathrm{U}(n-1)$.

Let $\{\theta^{\#}, J\theta^{\#}\}^\perp$ be the subbundle whose vectors are perpendicular to $\{\theta^{\#}, J\theta^{\#}\}$ with respect to g. As $\{\theta^{\#}, J\theta^{\#}\}^\perp$ coincides with the distribution $\mathrm{Null}\ \theta \cap \mathrm{Null}\ \theta\circ J$, it is invariant under J. By the above definition, f maps the subbundle $\{\theta^{\#}, J\theta^{\#}\}^\perp$onto itself and is holomorphic on it, *i.e.*, $f_*\circ J = J\circ f_*$.

On the other hand, since $f^*\theta = \theta, f^*(\theta\circ J) = \lambda\cdot(\theta\circ J)$, $f_*\theta^{\#} = \theta^{\#} + A, f_*J\theta^{\#} = \lambda\cdot J\theta^{\#} + B$ for some $A, B \in \{\theta^{\#}, J\theta^{\#}\}^\perp$. Hence TM decomposes into the direct sum $\{f_*\theta^{\#}, f_*J\theta^{\#}\} \oplus \{\theta^{\#}, J\theta^{\#}\}^\perp$. Define an almost complex structure J' on M simply to be

$$\begin{aligned} J'f_*\theta^{\#} &= f_*(J\theta^{\#}), \\ J'f_*(J\theta^{\#}) &= -f_*\theta^{\#}, \\ J'|\{\theta^{\#}, J\theta^{\#}\}^\perp &= J. \end{aligned}$$

If a vector $X = a\theta^{\#} + b(J\theta^{\#}) + V \in TM$ for $V \in \{\theta^{\#}, J\theta^{\#}\}^\perp$, then we can check that $f_*(JX) = J'f_*(X)$. Thus f is J'-holomorphic on M so that J' is a complex structure.

We put $\omega' = f^{-1*}\omega$. Since $f^{-1}{}_* \circ J' = J \circ f^{-1}{}_*$, it is easy to see that the two-form ω' is J'-invariant. Letting $\theta' = f^{-1*}\theta$, $d\omega' = \theta' \wedge \omega'$ such that $d\theta' = 0$. If we set $g'(X,Y) = \omega'(J'X,Y)$, then g' is J'-invariant l.c.K. metric on M. We can prove that $g'(f_*X, f_*Y) = g(X,Y)$. Hence f defined by (3.1) is a l.c.K. diffeomorphism of (M,g,J) onto (M,g',J'). We call such a transformation f a *Lee-Cauchy-Riemann* (*LCR*) *transformation* from one l.c.K. manifold (M,g,J) to another l.c.K. manifold (M,g',J').

If f' is a LCR transformation satisfying the equations that $f'^*\theta = \theta$, $f'^*(\theta\circ J) = \lambda' \cdot (\theta \circ J)$, $f'^*\theta^\alpha = \sqrt{\lambda'} \cdot \theta^\beta U'^\alpha_\beta + (\theta\circ J)\cdot v'^\alpha$, $f'^*\bar\theta^\alpha = \sqrt{\lambda'} \cdot \bar\theta^\beta_\alpha \overline{U'}^\alpha_\beta + (\theta\circ J)\cdot \overline{v'}^\alpha$, then it is easy to see that

$$
\begin{aligned}
&(f'\circ f)^*\theta = \theta, \quad (f'\circ f)^*(\theta\circ J) = (f^*\lambda' \cdot \lambda)\cdot(\theta\circ J),\\
&(f'\circ f)^*\theta^\alpha = \sqrt{f^*\lambda'\cdot\lambda}\cdot\theta^\beta W^\alpha_\beta + (\theta\circ J)\cdot w^\alpha, \qquad (3.2)\\
&(f'\circ f)^*\bar\theta^\alpha = \sqrt{f^*\lambda'\cdot\lambda}\cdot\bar\theta^\beta \overline{W}^\alpha_\beta + (\theta\circ J)\cdot \overline{w}^\alpha.
\end{aligned}
$$

The composition $f'\circ f$ is also a LCR transformation from (M,g,J) to another l.c.K. manifold. Denote by $\mathrm{Aut}_{LCR}(M,g,J,\theta)$ (or simply by $\mathrm{Aut}_{LCR}(M)$) the group of all Lee-Cauchy-Riemann transformations on a l.c.K. manifold (M,g,J) adapted to the Lee form θ.

Consider the subgroup G of $\mathrm{GL}(2n,\mathbb{R})$ consisting of the following elements:

$$
\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & v^\alpha & v^{\bar\alpha} \\ 0 & 0 & \sqrt{u}U^\alpha_\beta & 0 \\ 0 & 0 & 0 & \sqrt{u}U^{\bar\alpha}_{\bar\beta} \end{pmatrix} \mid u\in\mathbb{R}^+, v^\alpha\in\mathbb{C}, U^\alpha_\beta \in \mathrm{U}(m) \right\},
$$

where $m = n-1$. Let $G \to P \to M$ be the principal bundle of the G-structure consisting of the above coframes $\{\theta, \theta\circ J, \theta^\alpha, \theta^{\bar\alpha}\}$. If we note that G is isomorphic to the semidirect product $\mathbb{C}^n \rtimes (\mathrm{U}(m)\times\mathbb{R}^+)$, then the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{C}^n + \mathfrak{u}(m) + \mathbb{R}$. In particular, the matrix group $\mathfrak{g} \subset \mathfrak{gl}(2n,\mathbb{R})$ has no element of rank 1, *i.e.* it is *elliptic* (note that $\mathbb{C}^n$ is of infinite type, while $\mathfrak{u}(m)+\mathbb{R}$ is of order 2 (cf. [7]).) As M is assumed to be compact, the group of automorphisms $\mathcal{U}$ of P is a (finite dimensional) Lie group. Since $\mathrm{Aut}_{LCR}(M)$ is a closed subgroup of $\mathcal{U}$, $\mathrm{Aut}_{LCR}(M)$ is a Lie group. The compactness of the group $Aut_{LCR}(M)$, on a compact l.c.K. manifold (M,g,J) will be discussed in a subsequent paper. By Proposition 2.1, each element of $Aut_{l.c.K.}(M)$ satisfies $f^*\theta = \theta$, $f^*(\theta\circ J) = (\theta\circ J)$ and $f^*\omega = \omega$ with respect to some specific coframes $\{\theta,\theta\circ J,\theta^\alpha,\theta^{\bar\alpha}\}$. The fact that $f^*\omega = \omega$ implies that $f^*\theta^\alpha = \theta^\beta U^\alpha_\beta$, $f^*\bar\theta^\alpha = \bar\theta^\beta\overline{U}^\alpha_\beta$ for some positive

matrix $U^\alpha_\beta \in \mathrm{U}(n-1)$. Note that the elements of $Aut_{l.c.K.}(M)$ are also viewed as LCR transformations.

When a noncompact Lee-Cauchy-Riemann transformation subgroup of $Aut_{LCR}(M)$ acts on a compact l.c.K. mamifold M, we prove a similar property to the noncompact CR-action on a compact CR-manifold. The next result, the proof of which will be mostly situated in the realm of CR-geometry, characterizes the Hopf manifolds, up to biholomorphism, among the compact locally conformally Kähler manifolds.

Theorem C. *Let (M, g, J) be a compact locally conformally Kähler manifold with Lee form θ. Suppose that there exists a closed noncompact subgroup $\mathbb{R} = \{\phi_t\}_{t\in\mathbb{R}}$ of $Aut_{LCR}(M)$ satisfying that*

1. $(\phi_t)_*(\theta^\#) = \theta^\#$.
2. $(\phi_t)_*$ *preserves the distribution* $\{\theta^\sharp, (\theta\circ J)^\sharp\}^\perp$ *and is holomorphic on it.*

Then the following hold:
(i) *If θ is parallel and the Lee field $\theta^\#$ generates a circle S^1, then (M, g, J) is holomorphically isometric to a finite quotient of a Hopf manifold $S^1 \times S^{2n-1}/\Gamma$ where $\Gamma \subset S^1 \times \mathrm{U}(n-1)$.*
(ii) *If the Lee field $\theta^\#$ generates a purely conformal circle S^1, then (M, g, J) is holomorphically conformal to a finite quotient $S^1 \times S^{2n-1}/\Gamma$ $(\Gamma \subset S^1 \times \mathrm{U}(n-1))$.*

From this theorem, we obtain the following:

Corollary D. *Let (M, g, J) be a compact locally conformally Kähler manifold which admits a closed subgroup $\mathbb{C}^*$ of Lee-Cauchy-Riemann transformations adapted to the Lee form θ. Then,*
(1) *If θ is parallel and a subgroup S^1 of $\mathbb{C}^*$ induces the Lee field $\theta^\#$, then some finite cover of M is holomorphically isometric with a Hopf manifold $S^1 \times S^{2n-1}$.*
In general,
(2) *If there exists a purely conformal circle S^1 in $\mathbb{C}^*$ which induces the Lee field $\theta^\#$, then M is holomorphically conformal to a finite quotient of a Hopf manifold $S^1 \times S^{2n-1}$.*

Proof of theorem C.

Let (M, g, J) be a compact locally conformally Kähler manifold satisfying the conditions **1, 2**. We work with the same l.c.K. metric g on (M, J) for the case **(i)**, while applying Theorem A for the case **(ii)** first of all, there is a l.c.K. metric $\hat{g}$ with parallel Lee form which is conformal to g. (As a consequence, note that the Lee field $\theta^\#$ of g becomes Killing with respect to $\hat{g}$.) Then we work with $(M, \hat{g}, J)$ to show that it is holomorphically isometric with a finite quotient of a Hopf manifold.

(As a matter of fact, (M, g, J) is holomorphically conformal to a finite quotient of a Hopf manifold.).

We retain the same notation ϕ_t for the lift of $\mathbb{R} =< \phi_t >$ to the universal covering $\tilde{M}$. Suppose that the hypothesis of **(i)** is satisfied. Let $\hat{\varphi}_t$ be an element of S^1 generated by the Lee field $\theta^{\#}$. By **1**, each ϕ_t commutes with every element $\hat{\varphi}_t$. Then each lift ϕ_t to $\tilde{M}$ also commutes with the elements of the lift $\mathbb{R}^+$ of S^1. Let $\mathbb{R}^+ = \{\varphi_t\}_{t\in\mathbb{R}}$ be a 1-parameter group. (See **(II)** of Corollary B.) Then

$$(*) \qquad \phi_t \circ \varphi_s = \varphi_s \circ \phi_t \quad (t, s \in \mathbb{R}).$$

On the other hand, if the hypothesis of **(ii)** is satisfied, then as in the proof of Theorem A, the S^1-action lifts to a $\mathbb{R}^+(= \{\varphi_t\}_{t\in\mathbb{R}})$-action which satisfies $(*)$.

In each case, let ξ be a vector field on $\tilde{M}$ induced by the $\mathbb{R}^+$-action and note that $p_*\xi = \theta^{\#}$. By $(*)$, the projection $\mathbb{R}^+ \rightarrow \tilde{M} \xrightarrow{\pi} W$ induces a $\mathbb{R} = \{\phi_t\}$-action on W. We have the following commutative diagram:

$$(3.3) \qquad \begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \pi_1(M) & \longrightarrow & Q \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^+ & \longrightarrow & (\mathbb{R}^+ \times \mathbb{R}, \tilde{M}) & \xrightarrow{\pi} & (\mathbb{R}, W) \\
\downarrow & & \downarrow p & & \downarrow p \\
S^1 & \longrightarrow & (S^1 \times \mathbb{R}, M) & \xrightarrow{\pi^*} & (\mathbb{R}, M^*)
\end{array}$$

where $M^* = M/S^1$ and in the top line of the diagram, $Q = \pi_1(M)/\mathbb{Z}$. By **2**, each ϕ_t preserves the distribution $\{\theta^\sharp, (\theta \circ J)^\sharp\}^\perp$. We observe that $p_*\xi = \theta^{\#}$, $p_*(J\xi) = J\theta^{\#} = -(\theta \circ J)^{\#}$, hence p maps $\{\xi, J\xi\}^\perp$ isomorphically onto $\{\theta^\sharp, (\theta \circ J)^\sharp\}^\perp$ at each point of M. Since $\pi_* : \{\xi, J\xi\}^\perp \rightarrow \text{Null}\ \eta$ is J-isomorphic, the induced $\mathbb{R}$-action $< \phi_t >$ on W preserves Null η on which it is holomorphic, *i.e.*, $\phi_t^*\eta = \lambda_t \cdot \eta$ for some map $\lambda_t : W \rightarrow \mathbb{R}^+$, and $\phi_{t*} \circ J = J \circ \phi_{t*}$. Hence $\mathbb{R} = \{\phi_t\}$ is a closed subgroup of CR-transformations of (W, η, J). Note that the contact structure η of the Sasakian manifold W gives a strictly pseudoconvex, pseudo-Hermitian structure and the quotient space W/Q inherits a CR-structure from (η, J) by the property **1** of (2.6). Then the CR-action of W induces a CR-action of $\mathbb{R}$ on W/Q.

When W/Q happens to be a compact smooth manifold, we can apply Webster's theorem in [14] to yield that W/Q is spherical. But in general, W/Q is a compact orbifold. Even so, we can still show that it is spherical. The key step will be, as in [14], to prove:

Proposition 3.1. *The CR-manifold $(W, \{\mathrm{Null}\ \eta, J\})$ is spherical (or CR-flat, in another terminology, meaning that its Chern-Moser curvature tensor S vanishes identically on W).*

Proof. Suppose W is not spherical. Then the set $V = \{x \in W \mid S_x \neq 0\}$ is a non-empty, open subset of W. As S is a CR invariant (see *e.g.* [14]), the set V is preserved by any CR-automorphism of W. If a contact form η is replaced by $\eta' = u \cdot \eta$, then the norm of the Chern-Moser tensor with respect to the Levi form associated to η satisfies the equality $\|S\|_\eta = u \cdot \|S\|_{\eta'}$ (cf. [15]). Choose $u = \|S\|_\eta$ on V. Then we obtain a pseudo-Hermitian structure (η', J) defined on V such that

$$\|S\|_{\eta'} = 1 \text{ on } V. \tag{3.4}$$

If $f \in Aut_{CR}(W)$, then $f^*\eta' = \lambda \cdot \eta'$. The above relation shows that $\|S\|_{\eta'}(x) = \lambda \cdot \|S\|_{f^*\eta'}(x) = \lambda \cdot \|S\|_{\eta'}(f(x))$. By (3.4), $\lambda = 1$ on V. Then we can check that for each element f of $Aut_{CR}(W)$, the transformation (3.1) can be reduced to the following:

$$\begin{aligned} f^*\eta' &= \eta' \\ f^*\theta'^\alpha &= \theta'^\beta U^\alpha_\beta \\ f^*\overline{\theta}'^\alpha &= \overline{\theta}'^\beta \overline{U}^\alpha_\beta \\ d\eta' &= \sum_{\alpha,\beta} \theta'^\alpha \wedge \overline{\theta}'^\beta. \end{aligned}$$

Thus the bundle of such coframes $\{\eta', \eta' \circ J, \theta'^\alpha, \overline{\theta'}^\alpha\}_{\alpha=1,\cdots,n-1}$ with $d\eta' = \sum_{\alpha,\beta} \theta'^\alpha \wedge \overline{\theta}'^\beta$ gives rise to a principal bundle, restricted to V:

$$\mathrm{U}(n) \rightarrow P' \xrightarrow{q} V, \tag{3.5}$$

for which we note that

$$Aut_{CR}(W) \text{ is a } \textit{closed} \text{ subgroup of } Aut_{U(n)}(V). \tag{3.6}$$

On the other hand, as $\mathrm{U}(n)$ is of order 1 (as a subgroup of $\mathrm{O}(2n)$), the manifold P' has a $\{1\}$-structure. Hence, by Theorem 3.2 in [7], for any fixed $u' \in P'$, the orbit map $Aut_{U(n)}(P') \rightarrow P'$ is a proper embedding, and the orbit $Aut_{U(n)}(P') \cdot u'$ is closed in P'. As any automorphism of W produces, by differentiation, an automorphism of P', we may consider the subgroup $Aut_{U(n)}(V)_* = \{df \mid f \in Aut_{U(n)}(V)\}$. Restricting to CR-automorphisms of W and using (3.6), we derive that the orbit $Aut_{CR}(W)_* \cdot u'$ is closed in P' and that $Aut_{CR}(W)_* \cdot u'$ is homeomorphic to $Aut_{CR}(W)$. Now recall that we have a closed subgroup $\mathbb{R}$ in $Aut_{CR}(W)$. Denote with $\mathbb{R}_*$ its lift to P' ($\mathbb{R}_*$ contains the differentials of the flows associated to the $\mathbb{R}$-action). Hence, $\mathbb{R}_* \cdot u'$ is closed in P'.

Now, if we can prove that this orbit is contained in a certain compact subset $K' \subset P'$, we see that $\mathbb{R}_* \cdot u'$ has to be compact, in contradiction with it being homeomorphic with $\mathbb{R}$ by the proper embedding.

To construct K', set $x' = q(u') \in V$ so that $q(\mathbb{R}_* \cdot u') = \mathbb{R} \cdot x'$. From the diagram (3.3), p is a quotient (continuous) map and W is locally compact (as a manifold), M^* is compact, hence we may find a connected, compact subset $C \subset W$ such that $x' \in C'$ and $p(C) = M^*$. Hence $W = \underset{\alpha \in Q}{\cup} \alpha \cdot C$. In particular, as $\mathbb{R} \cdot x' \subset V$, we obtain $\mathbb{R} \cdot x' \subset \underset{\alpha \in Q}{\cup} \alpha \cdot C$. But Q acts properly discontinuously, so only a finite number of translated $\alpha \cdot C$ meet C. Since $\mathbb{R} \cdot x'$ is connected, we may write $\mathbb{R} \cdot x' \subset C' := \overset{k}{\underset{i=1}{\cup}} \alpha_i \cdot C$ and note that C' is compact. Hence, the closure $\overline{\mathbb{R} \cdot x'}$ is compact. But *a priori,* it might exit V. We prove that this is not the case and, in fact, $\overline{\mathbb{R} \cdot x'} \subset V$ following [14]. Then the inverse image $K' = q^{-1}(\overline{\mathbb{R} \cdot x'})$ of the bundle (3.5) is the one desired.

To this end, let A be the vector field on W induced by the considered $\mathbb{R}$-action $< \phi_t >$. We first prove:

$$\eta'(A) \text{ does not vanish identically on } V. \tag{3.7}$$

Indeed, by absurd, if $\eta'(A) = 0$ on V, then $A \in \text{Null}\,\eta'$ on V. For any $X \in \text{Null}\,\eta'$, as A an infinitesimal contact transformation, we have $0 = (\mathcal{L}_A\eta')(JX) = A(\eta'(JX)) - \eta'([A, JX]) = 2d\eta'(A, JX)$, in contradiction with (V, η') being strictly pseudo-convex, its Levi form $d\eta'$ is positive definite, in particular non-degenerate. If we note that x' $(= q(u'))$ can be chosen arbitrary in V, we may suppose $\eta'(A_{x'}) = d$ for some suitable, fixed $d \in \mathbb{R}^*$.

We consider the non-empty set $D = \{x \in V\ ;\ \eta'(A_x) = d\}$ and show that:

$$D \text{ is closed in } W. \tag{3.8}$$

Here is a simple argument of general topology. Observe first that, by $\eta' = u \cdot \eta$ with $u = \|S\|_\eta$, $u(x)$ tends to 0 when x approaches the boundary of V in W. Now let $x \in \overline{D}$ and choose a sequence $\{x_n\} \subset D$ which converges to x. We have $d = \eta'_{x_n}(A_{x_n}) = u(x_n) \cdot \eta_{x_n}(A_{x_n})$. But $\eta_{x_n}(A_{x_n}) \to \eta_x(A_x) < \infty$ as $x_n \to x$, hence $u(x_n) \not\to 0$. This means that x is not a boundary point of V, thus $x \in V$. This implies $\eta'(A_x) = \lim \eta'(A_{x_n}) = d$, yielding $x \in D$ and proving (3.8).

Since $f^*\eta' = \eta'$ for $f \in Aut_{CR}(W)$, the equality $\eta'(\phi_{t*}A_{x'}) = \eta'(A_{x'})$ $= d$ implies $\mathbb{R} \cdot x' \subset D$. Obviously, $\overline{\mathbb{R} \cdot x'} \subset V$. This ends the proof of Proposition 3.1. ∎

Now, in order to finish the proof of Theorem C, we analyse the following diagram:

$$\begin{tikzcd} Q \arrow[d] & \\ (\mathbb{R}, W) \arrow[r, "(\rho, dev)"] \arrow[d] & (PU(n,1), S^{2n-1}) \\ (\mathbb{R}, W/Q) & \end{tikzcd}$$

where in the horizontal line we have the developing pair (see [8] for the definition) of $(\mathbb{R}, W)$ and $PU(n,1)$ is the group of CR-automorphisms of the standard sphere viewed as boundary of the complex hyperbolic space (cf. [5]). If W is not CR-equivalent with S^{2n-1}, then $X := dev(W)$ may consist of $S^{2n-1} - \{\infty\}$ or $S^{2n-1} - \{0, \infty\}$ (see Theorem 4.4 in *loc. cit.*). We show that these two cases lead to contradiction. Indeed, if by absurd either of the two cases occur, we may pull back by dev their metrics to W obtaining a Q-invariant metric. This one descends to a complete metric of the compact orbifold W/Q. We take the associate distance function (which is continuous) and we can lift it back to W to a complete distance (all this was necessary because W is not compact!). It follows that dev is a covering map, hence a diffeomorphism on the image. As a consequence, a finitely generated subgroup Q is isomorphic with $\rho(Q)$ in $PU(n,1)$. According to Selberg's result (cf. [10], Corollary 6.14) there exists a subgroup Q' of finite index in Q which is torsion free. Hence, W/Q' is a compact manifold. Then the group of CR automorphisms of W/Q' is compact because it is a closed subgroup of the group of isometries W/Q' induced by the metric from W. Still, we know that it must contain a closed subgroup $\mathbb{R}$. This contradiction shows that dev maps W onto S^{2n-1} CR-diffeomorphically. This finishes the proof.

Remark 3.1. The $\mathbb{R}$-action on S^{2n-1} is characterized as either loxodromic ($= \mathbb{R}^+$) or parabolic ($= \mathcal{R}$) for which $\mathbb{R}^+$ has exactly two fixed points $\{0, \infty\}$ or $\mathcal{R}$ has the unique fixed point $\{\infty\}$ on S^{2n-1}. Moreover, since $\mathbb{R}$ centralizes Q, it implies either $\mathbb{R} \times Q \subset \mathcal{R} \times U(n)$ or $\mathbb{R} \times Q \subset \mathbb{R}^+ \times U(n)$ where Q is a finite subgroup by properness. As $W/Q(\approx S^{2n-1}/Q)$ is an orbifold, such a finite subgroup may exist, contrary to the case that W/Q is a compact smooth CR-*manifold*.

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